A Controllability Synthesis Problem for Dynamic Multi-Agent Systems with Linear High-Order Protocol

Ning Cai, Junwei Cao, and M Junaid Khan

Abstract: In this paper, a problem on controllability synthesis for dynamic multi-agent systems with high-order linear protocol is addressed. A particular scenario is considered, in which a given system should be adjusted to be controllable by the input information upon the created leaders, with optional communication links from the leaders to each follower vertex in the graph. It is shown that a single leader is always sufficient to satisfy the requirement for controllability of any given graph topology, so long as there are proper links between the leader and the followers.

Keywords: Controllability, multi-agent systems, leader, follower, graph topology.

1. INTRODUCTION

The controllability of a dynamic system reflects the extent of influence from input to the motion of the system. Observability and controllability are dual alternatives. Integrated with stability, other notions such as stabilizability and detectability can be induced by these concepts. They form the theoretical foundation for most of the system synthesis problems in modern control theory, e.g. pole assignment, dynamic compensator, and optimal regulator. Thus, controllability is an extraordinarily important concept.

Controllability of composite systems has been extensively studied since the 1960s. Scholars attempted to propose criteria for checking controllability [1-6]. During recent years, as multi-agent systems greatly hold interest of scholars in the field of control theory, only few of them have noticed the controllability problems. Tanner [7] initially defined the concept of graph controllability under the leader-follower framework, based on a partition of Laplacian matrices. His definition has been accepted as a standard. Ji *et al.* [8] proposed certain sufficient conditions for controllability of graph topologies, on the basis of the algebraic characteristics of related matrices. They compared the endeavor to control the overall multi-agent system through a few leaders to shepherding. Rahmani *et al.* [9] extended the work of Ji *et al.* in [8], concerning the effect to controllability induced by topological symmetry. Cai *et al.* [10-12] studied formation controllability of high-order systems and discussed approaches for controllability improvement of them, based on a generalized controllability canonical form of dynamic systems. Liu *et al.* [13] addressed the controllability of discrete-time multi-agent systems with switching graph topologies, applying their previous results on controllability of linear switching systems. Wang *et al.* [14] discussed the influence on controllability from different leader assignments and communication links. Ji *et al.* [15] proposed conditions for graph controllability, which are analogous to the early results in [5]. Liu *et al.* [16] attempted to explore the criterion of structural controllability for large-scale complex networks in a multiple disciplinary background. Some recent relevant works include [17-21] and the references therein.

In this paper, a controllability synthesis problem of dynamic multi-agent systems is addressed, which is an extension of our previous series of works on controllability of multi-agent systems. A specific scenario is considered, in which a given graph topology should be converted by creating new leaders and connecting them with the existing followers, with optional information links from the leaders to each follower vertex. This paper will theoretically indicate that a single leader is always sufficient to satisfy the requirement for controllability of any given graph topology.

The motivation of this paper arises from the consideration of synthesizing some multi-agent systems in engineering, e.g. the multi-agent supporting systems [10, 22] and multi-aircraft systems [23, 24]. In many multi-agent systems, the dynamics of subsystems are generally intrinsic and unalterable, whereas the communication links are adjustable. Usually a prescribed formation is expected to be achieved. Evidently, sufficient controllability for the states of agents in the overall system is indispensable. Since the main concern of this paper is on the controllability of graph topologies in general, the discussions could also potentially be referential for diverse composite systems with appropriate protocols, such as certain process systems [5], economic systems [25], and Hopfield neural networks [26].

Manuscript received August 3rd, 2013; revised November 6th, 2013 and February 19th, 2014; accepted April 18th, 2014.

This work was supported by National Natural Science Foundation of China (Grants 61263002, 61374054, & 61174067).

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The rest of this paper is organized as follows. Section 2 formulates the multi-agent system model and the controllability synthesis problem. Section 3 follows to provide the main theoretical results. A numerical example is shown in Section 4 to verify the theory. Finally, Section 5 contains the concluding remarks.

2. PROBLEM FORMULATION

2.1. High-order multi-agent systems

The high-order multi-agent system model to be considered comprises N agents, each with state vector $x_i \in R^d$. The dynamics are depicted by

$$\dot{x}_i = F\sum_{j=1}^{N} w_{ij}(x_j - x_i) \quad (i = 1,...,N) \tag{1}$$

with F being a matrix with appropriate dimension, and $w_{ij} \in R$ the edge or arc weight of the graph topology, representing the strength of communication link between two neighboring vertices. A time-invariant weighted graph can be uniquely determined by its adjacency matrix, which is denoted by $W = [w_{ij}]$ here.

All the N agents in the original system are follower agents since there are no entries for any external input to control them. In order to improve the controllability of the multi-agent system, new leader vertices should join the graph topology.

Suppose N_l leaders are created to control the graph, which have analogous identity to the follower vertices but can provide entries to receive external input. With the leaders joined, the dynamics of the multi-agent system can now be depicted by:

$$\dot{x}_i = F\sum_{j=1}^{N+N_l} w_{ij}(x_j - x_i) + Bu_i \quad (i = 1,...,N, N+1,...,N_l) \tag{2}$$

where $B \in R^{d\times m}$, and $u_i \in R^m$ is the input vector for agent i. The first N agents are followers, without entries for external input, i.e. $u_i(t) \equiv 0$ ($i = 1,2,...,N$), whereas the last N_l agents are leaders, actuated by certain $u_i(t)$ not being zero identically.

If all state vectors of vertices are stacked together, then the overall state matrix of the system (2) is

$$X = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1(N+N_l)} \\ x_{21} & x_{22} & \cdots & x_{2(N+N_l)} \\ \vdots & \vdots & & \vdots \\ x_{d1} & x_{d2} & \cdots & x_{d(N+N_l)} \end{bmatrix}$$

The dynamics of the system can be described by the following matrix state equation:

$$\dot{X} = -FXL + BU \tag{3}$$

with L the Laplacian matrix of the graph topology. The relationship between the adjacency and the Laplacian matrices of a graph topology is:

$$L = diag(W\phi) - W \tag{4}$$

where ϕ denotes a vector with all entries being 1.

3.2. Controllability synthesis problem

For the multi-agent system with leaders (2), partition L to discriminate between the followers and the leaders:

$$L = \begin{bmatrix} L_{ff} & L_{fl} \\ L_{lf} & L_{ll} \end{bmatrix} \tag{5}$$

where $L_{ff} \in R^{N\times N}$ indicates the arcs in the graph among the followers; $L_{fl} \in R^{N\times N_l}$ the arcs from the leaders to the followers; $L_{lf} \in R^{N_l\times N}$ the arcs from the followers to the leaders; and $L_{ll} \in R^{N_l\times N_l}$.

Definition 1: (Graph Controllability [10], [15]) With the last N_l agents as the leaders and a partitioned form of Laplacian matrix L as (5), the graph topology is controllable if and only if (L_{ff}, L_{fl}) is controllable.

Remark 2: Definition 1 was originally proposed by Tanner [7] and has become a standard accepted by many scholars concerning controllability problems of multi-agent systems. According to this definition, the controllability of a graph is determined not only by its topology, but also by the specific leader-follower configuration.

With the criterion given by the following lemma, complete controllability of an LTI multi-agent system with high-order linear protocol can be checked.

Lemma 1 [10], [12]: The LTI multi-agent system (2) is completely controllable if and only if the two conditions below are simultaneously satisfied:

1) The graph topology is controllable;
2) The matrix pair (F, B) is controllable.

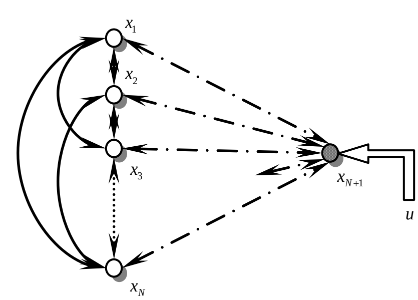

Fig. 1. Configuration of multi-agent system with single leader.

The following assumption is assumed to be satisfied throughout this paper. Under this assumption, the rest of the controllability problem of (2) is equivalent to the controllability of graph, i.e. controllability of the matrix pair (L_{ff}, L_{fl}).

Assumption 1: (F, B) is controllable.

The purpose of the main problem concerned in this paper is to build appropriate communication links between the leaders and the followers and to determine the weights of corresponding arcs in the topology so that a controllable graph can be achieved. The configuration of such a system is illustrated in Fig. 1, with the dash-dotted arrows representing the arcs to be designed.

To sum up, the problem to be handled here is: With given $W_{ff} \in R^{N\times N}$, how can one design a proper W_{fl} such that (L_{ff}, L_{fl}) is controllable.

Remark 3: Note that the number of columns in L_{fl} stands for the number of leaders.

In the next section, it will be shown that a single leader is sufficient to satisfy the requirement of controllability for any graph topology, if only with proper $W_{fl} \in R^{N\times 1}$.

3. CONTROLLABILITY SYNTHESIS BY A SINGLE LEADER

If without any leader, (4) and (5) naturally yield that

$$L_{ff} = diag(W_{ff}\phi) - W_{ff}$$

which is determined merely by W_{ff}. If a leader is created, L_{ff} will be affected by the leader even if W_{ff} is fixed:

$$L_{ff} = diag(W_{ff}\phi) - W_{ff} + diag(W_{fl}) \tag{6}$$

Besides,

$$L_{fl} = -W_{fl} \tag{7}$$

For general LTI systems taking the form

$$\dot{x} = Ax + Bu$$

if $A \in R^{n\times n}$ is given and $B \in R^{n\times m}$ is optional, then sometimes a single input variable is insufficient for controllability. The following two lemmas clearly manifest this phenomenon.

Definition 2: (Derogatory Matrix [27]) A square matrix $A \in R^{n\times n}$ is nonderogatory if the degree of its minimal polynomial is n, otherwise it is derogatory.

Lemma 2 [11]: For a given $A \in R^{n\times n}$, there exists a column vector $b \in R^{n\times 1}$ such that (A,b) is controllable if and only if *A* is nonderogatory.

Lemma 3 [11]: If the matrix $A \in R^{n\times n}$ in (A,b) is derogatory, then at least $\mu(A)$ input variables are required for (A,b) to be controllable, where $\mu(A)$ denotes the maximum geometric multiplicity of eigenvalues of A.

Remark 4: Lemmas 2 & 3 implies a fact that with a given A, it could be impossible for the matrix pair (A,b) to be controllable if there is only one single input variable.

Nonetheless, on the contrary, it is an interesting and meaningful fact that for any given multi-agent system with linear protocol, a single leader is always sufficient for controllability.

Theorem 1: Suppose that there are N followers with given information flow configuration $W_{ff} = [w_{ij}]$ among them, then there exists a leader with proper information links $W_{fl} = [w_{fl}^{(i)}] \in R^{N\times 1}$ such that the graph topology is controllable.

Proof: Consider vector

$$W_{fl} = [w_{fl}^{(i)}] = \left[i \times \alpha\right] \quad (i = 1, 2, \cdots, N)$$

where α is a positive real value. It will be proved that (L_{ff}, L_{fl}) is controllable if α is sufficiently large.

According to Gersgorin Disk Theorem [27], the spectrum of the matrix

$$L_{ff} = diag(W_{ff}\phi) - W_{ff} + diag(W_{fl})$$

locates inside the union of the following Gersgorin Disks:

$$C_i\left(w_{fl}^{(i)}\right) = \left\{ s \in C \,\middle|\, \left|s + w_{fl}^{(i)} - w_{ii}\right| \le \sum_{j=1, j\ne i}^{N} \left|w_{ij}\right| \right\}$$

If the value of α is sufficiently large such that

$$\max_{1\le i\le N-1} \left\{ \sum_{j=1}^{N} \left|w_{ij}\right| + \sum_{j=1}^{N} \left|w_{(i+1)j}\right| \right\} < \alpha$$

then

$$i\alpha + \left|w_{ii}\right| + \sum_{j=1, j\ne i}^{N} \left|w_{ij}\right| < (i+1)\alpha - \left|w_{(i+1)(i+1)}\right| - \sum_{j=1, j\ne i+1}^{N} \left|w_{(i+1)j}\right|$$

$$(i = 1, 2, \cdots, N-1)$$

and it follows that

$$\left|w_{ii} - w_{fl}^{(i)}\right| + \sum_{j=1, j\ne i}^{N} \left|w_{ij}\right| < \left|w_{(i+1)(i+1)} - w_{fl}^{(i+1)}\right| - \sum_{j=1, j\ne (i+1)}^{N} \left|w_{(i+1)j}\right|$$

$$(i = 1, 2, \cdots, N-1)$$

Notice that there are no intersections among these Gersgorin disks $C_i\left(w_{fl}^{(i)}\right)$ $(i = 1, 2, \cdots, N-1)$. Therefore, all the eigenvalues λ_i $(i = 1, 2, \cdots, N-1)$ are different, each locating inside one Gersgorin disk $C_i\left(w_{fl}^{(i)}\right)$ $(i = 1, 2, \cdots, N-1)$. The eigenvalues can be expressed as

$$\lambda_i = \hat{w}_i - w_{fl}^{(i)} = \hat{w}_i - i\alpha \quad (i = 1, 2, \cdots, N-1)$$

with

$$\left|\hat{w}_i\right| \le \sum_{j=1}^{N} \left|w_{ij}\right|$$

Assume that the matrix pair (L_{ff}, L_{fl}) is uncontrollable, then according to PBH test, there exist eigenvalues λ_k $\left(k \in \{1, 2, \cdots, N\}\right)$ of

$$diag(W_{ff}\phi) - W_{ff} + diag(W_{fl})$$

such that the rows in matrix

$$\left[diag(\left[\lambda_k + w_{fl}^{(i)}\right]) - W_{ff} \quad -W_{fl} \right] \tag{8}$$

are linearly dependent. (8) can be transformed by rearrangement of the rows and columns without change of its rank, and the result is (9)[i], where the first $N-1$ columns of matrix $W_{ff}^{(k)}$ are derived by rearrangement of the rows in the submatrix comprising the columns of with indexes $1, 2, \cdots, k-1, k+1, \cdots, N$, and the N th column is zero. None of the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_N$ belong to the matrix $W_{ff}^{(k)}$, and $W_{ff}^{(k)}$ and $-W_{fl}$ have no common elements. We know that the rows in (9) is also linearly dependent, thus there exists a vector η

satisfying $\|\eta\| = \sqrt{\sum_{i=1}^{N} |\eta|^2} = 1$ such that (10)[ii] holds. (10) can be simplified into

$$\Psi(\alpha)\eta = -W_{ff}^{(k)}\eta$$

with (11)[iii]. It is evident that $\Psi(\alpha)$ is nonsingular when the value of α is sufficiently large. Consequently,

$$\eta = -\Psi^{-1}(\alpha) W_{ff}^{(k)} \eta$$

Suppose that

$$\Psi^{-1}(\alpha) = [\psi_{ij}]$$

then it is easy to know that

$$\psi_{ii}(\alpha) = \begin{cases} \dfrac{1}{(k-i)\alpha + \hat{a}_i} & (i = 1, 2, \cdots, k-1) \\ \dfrac{1}{(k-i-1)\alpha + \hat{w}_{i+1}} & (i = k, k+1, \cdots, N-1) \\ \dfrac{1}{k\alpha} & (i = N) \end{cases}$$

$$\psi_{iN}(\alpha) = \begin{cases} \dfrac{i}{[(k-i)\alpha + \hat{w}_i]k} & (i = 1, 2, \cdots, k-1) \\ \dfrac{i+1}{[(k-i-1)\alpha + \hat{w}_{i+1}]k} & (i = k, k+1, \cdots, N-1) \end{cases}$$

$$\psi_{ij}(\alpha) = 0 \quad (i = 1, 2, \cdots, N;\ j = 1, 2, \cdots, N-1; i \neq j)$$

When α is sufficiently large,

$$|\psi_{ij}(\alpha)| \le \frac{\beta_{ij}}{\alpha} \quad (i, j = 1, 2, \cdots, N)$$

with $\beta_{ij} \ge 0$ $(i, j = 1, 2, \cdots, N)$ being independent of α. Therefore, when α is sufficiently large,

$$\|\eta\| = \|\Psi^{-1}(\alpha) W_{ff}^{(k)} \eta\| < 1$$

This contradicts the assumption that $\|\eta\| = 1$. So the conclusion can be drawn that the pair of matrices

$$\left(L_{ff} = diag(W_{ff}\phi) - W_{ff} + diag(W_{fl}), L_{fl} = -W_{fl} \right)$$

must be controllable, as long as α is sufficiently large. □

Note that the proof of Theorem 1 also provides a convenient method to construct a proper series of communication links from the leader to the followers, i.e. $W_{fl} = [i \times \alpha]$ $(i = 1, 2, \cdots, N)$, with α a positive value.

4. NUMERICAL EXAMPLE

Suppose that the original multi-agent system without leader has four agents, with the follower vertices indexed 1~4 and

$$W_{ff} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

W_{ff} is evidently a derogatory matrix, so according to Lemmas 2 and 3, there does not exist any vector $W_{fl} \in R^{4\times1}$ such that the matrix pair (W_{ff}, W_{fl}) is controllable. In other words, it is impossible to obtain a controllable dynamic system $\dot{x} = W_{ff}x + W_{fl}u$ with only one input variable, i.e. u is a scalar.

In order to improve the controllability of the multi-agent system, create one leader indexed 5. According to Theorem 1, let $W_{fl} = [1 \quad 2 \quad 3 \quad 4]^T$ with $\alpha = 1$, then

$$L_{fl} = -[1 \quad 2 \quad 3 \quad 4]^T$$

$$L_{ff} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

It is easy to verify that (L_{ff}, L_{fl}) is controllable, so is the graph with the leader. Thus, $W_{fl} = [1 \quad 2 \quad 3 \quad 4]^T$ is appropriate. The configuration of the resulting controllable graph is illustrated in Fig.2.

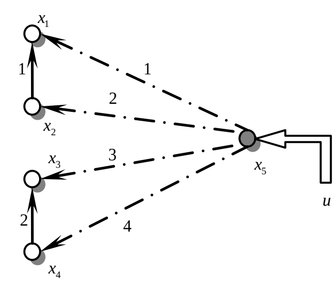

Fig. 2. Instance of resulted controllable graph topology.

5. CONCLUSION

A controllability synthesis problem of multi-agent systems with high-order LTI dynamic protocol is discussed. The main objective of the problem concerned is to control the states of agents in a system via some leaders. The number of leaders in a graph is analogous to the number of input variables in a dynamic system. It has been shown that, a single leader is always sufficient to meet the requirement for controllability, if only with appropriate communication links starting off the leader. The suggestion and proof of this paper is currently obvious but conservative, which could be compensated by applying the refined eigenvalue perturbation theory for matrices. And then, the general multi-agent systems with self-dynamics could also be considered.

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i
$$\left[\begin{bmatrix} \lambda_k + w_{fl}^{(1)} & & & & & & & w_{fl}^{(1)} \\ & \lambda_k + w_{fl}^{(2)} & & & & & & w_{fl}^{(2)} \\ & & \ddots & & & & & \vdots \\ & & & \lambda_k + w_{fl}^{(k-1)} & & & & w_{fl}^{(k-1)} \\ & & & & \lambda_k + w_{fl}^{(k+1)} & & & w_{fl}^{(k+1)} \\ & & & & & \ddots & & \vdots \\ & & & & & & \lambda_k + w_{fl}^{(N)} & w_{fl}^{(N)} \\ & & & & & & & w_{fl}^{(k)} \end{bmatrix} + W_{ff}^{(k)} \qquad \begin{matrix} -w_{1k} \\ -w_{2k} \\ \vdots \\ -w_{(k-1)k} \\ -w_{(k+1)k} \\ \vdots \\ -w_{nk} \\ \lambda_k + w_{fl}^{(k)} - w_{kk} \end{matrix}\right] \tag{9}$$

ii
$$\begin{bmatrix} \lambda_k + w_{fl}^{(1)} & & & & & & & w_{fl}^{(1)} \\ & \lambda_k + w_{fl}^{(2)} & & & & & & w_{fl}^{(2)} \\ & & \ddots & & & & & \vdots \\ & & & \lambda_k + w_{fl}^{(k-1)} & & & & w_{fl}^{(k-1)} \\ & & & & \lambda_k + w_{fl}^{(k+1)} & & & w_{fl}^{(k+1)} \\ & & & & & \ddots & & \vdots \\ & & & & & & \lambda_k + w_{fl}^{(N)} & w_{fl}^{(N)} \\ & & & & & & & w_{fl}^{(k)} \end{bmatrix} \eta + W_{ff}^{(k)} \eta = 0 \tag{10}$$

iii
$$\Psi(\alpha) = \begin{bmatrix} (k-1)\alpha + \hat{w}_1 & & & & & & -\alpha \\ & (k-2)\alpha + \hat{w}_2 & & & & & -2\alpha \\ & & \ddots & & & & \vdots \\ & & & \alpha + \hat{w}_{k-1} & & & -(k-1)\alpha \\ & & & & -\alpha + \hat{w}_{k+1} & & -(k+1)\alpha \\ & & & & & \ddots & \vdots \\ & & & & & (k-N)\alpha + \hat{w}_n & -N\alpha \\ & & & & & & -k\alpha \end{bmatrix} \tag{11}$$